PIVOT TRANSFORMS

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Introduction

The pivot transform is described by Lawrence Edwards (Reference 1). The idea is as follows. Given a three dimensional homogeneous linear transformation **T**, any surface S may be regarded as an envelope of planes. The transformation moves these planes, and for any such plane α there is in general a point P about which it is momentarily turning, or which is instantaneously invariant. The point P is referred to as the *pivot point* of that plane wrt **T**. The locus of the pivot points for all tangent planes to S is referred to as the *pivot transform* of S. In particular those transformations with two real and and two complex conjugate eigenvalues have been studied. They were first applied by Edwards to describe the form of the gynaeceum of, for example, a rose. The success achieved encouraged further work which yielded embryo-like forms.

In this paper a brief summary of the basic mathematics is given, followed by its adaptation by the author to enable coloured shaded pictures simulating photographs to be constructed, some of which accompany the paper.

Preliminaries

T is a 4x4 matrix specifying a three dimensional projective transformation in homogeneous coordinates. If it has two real and two complex conjugate eigen values then the invariant tetrahedron of the transformation degenerates into one with two real invariant planes, two real invariant points and two real invariant lines. The remaining invariants are imaginary. If further the two real planes are parallel and horizontal, and one of the two lines is orthogonal to them, then the invariant curves of the transformation (referred to as *path curves*) are either egg shaped or vortex shaped according to the choice of eigenvalues. Figure 1 illustrates the tetrahedron and a typical egg path curve.

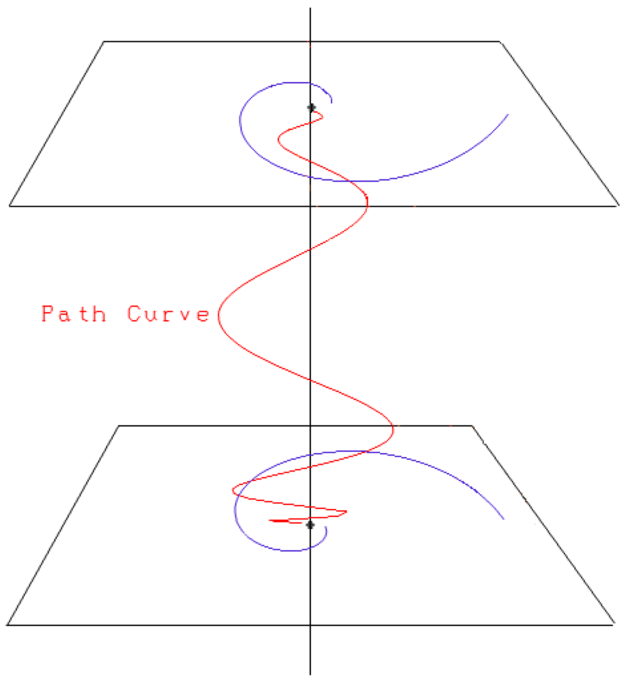

Figure 1

These invariant curves were first treated by Felix Klein, and each is a locus as follows. Given any point P (not an invariant point) application of **T** yields a new point P_1 which is the transform of P. Applying **T** to P_1 yields a third point P_2 and so on, and the locus of the P_j is a path curve of the system. The path curve is, strictly speaking, the invariant curve through P when the step size is infinitesimal. For a given **T** there is a unique path curve through every point of space.

We may express **T** in diagonal form as

$$\begin{bmatrix} \phi e^{i\theta} & & & \\ & \phi e^{-i\theta} & & \\ & & \Lambda_2 & \\ & & & \Lambda_3 \end{bmatrix}$$

where the Λ_j are the eigenvalues, but since Λ_0 and Λ_1 are complex conjugate we have shown them accordingly in terms of the radial expansion factor Φ and the rotation θ induced by the transformation. A general point z_j is such that z_0 is the horizontal homogeneous x-coordinate, z_1 the horizontal coordinate orthogonal to z_0, z_2 the vertical coordinate which is zero in the lower invariant plane, and z_3 the coordinate which is zero in the upper invariant plane. We use the symbol z to indicate that real points have complex coordinates when we express **T** in diagonal form. If z_j lies in the lower invariant plane then $z_2=0$. The radial distance r of such a point from the axis is calculated using the moduli $|z_j|$ of the coordinates i.e.

$$r=\sqrt{\left(\frac{|z_0|}{|z_3|}\right)^2+\left(\frac{|z_1|}{|z_3|}\right)^2}=\frac{\sqrt{|z_0|^2+|z_1|^2}}{|z_3|}$$

Transforming z_j by **T** gives $\{\Phi z_0 e^{i\theta}, \Phi z_1 e^{-i\theta}, 0, \Lambda_3 z_3\}$, and the radial distance r' is now

$$\frac{\sqrt{|\phi z_0 e^{i\theta}|^2+|\phi z_1 e^{-i\theta}|^2}}{\Lambda_3 |z_3|}=\frac{\phi\sqrt{|z_0|^2+|z_1|^2}}{\Lambda_3 |z_3|}=\frac{\phi}{\Lambda_3}r$$

so that the transform causes a fixed radial expansion Φ/Λ_3 for each angle θ turned through. This of course gives a logarithmic spiral, which has the general equation

$$r=ae^{\theta\cot\alpha}$$

and going from r_1 to r_2 we have

$$\frac{r_2}{r_1}=e^{(\theta_2-\theta_1)\cot\alpha}=\frac{\phi}{\Lambda_3}$$

where $(\theta_2-\theta_1)$ is our stepping angle θ. Hence

$$\tan\alpha=\frac{\theta}{\log\left(\frac{\phi}{\Lambda_3}\right)} \tag{0}$$

giving the characteristic angle α for all path curve spirals in the lower invariant plane; it is the constant angle between the radius vector and tangent at a point. We get a similar result in the upper invariant plane, replacing Λ_3 with Λ_2 as $z_3=0$ in that plane. We denote the constant angles of the spirals as α_2 and α_3. For an egg path curve the spirals in the two planes wind in opposite directions, and the path curve may be constructed from these spirals as shown below:

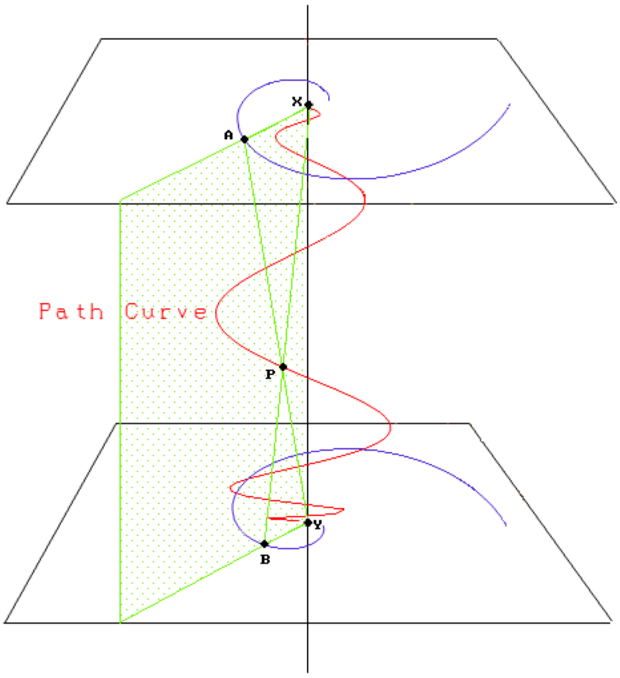

Figure 2

The plane XYAB shown rotates about the vertical axis XY. It intersects the top spiral in points one of which is shown as A. Likewise B is an intersection with the spiral in the lower invariant plane. The joins of XB and YA meet in a point P on the path curve. As the plane rotates A and B move round their spirals and P moves so as to describe the path curve.

At each point of the curve such as P there is an osculating plane which has triple contact with the curve i.e. is such that the curve lies in it at P. Dually, given any plane there is just one point in it at which a path curve of the system has triple contact, and this point is the *pivot point* of that plane for the particular transformation concerned.

To find such a pivot point, we note that the plane intersects the top and bottom planes in parallel lines each of which touches a logarithmic spiral in its plane.

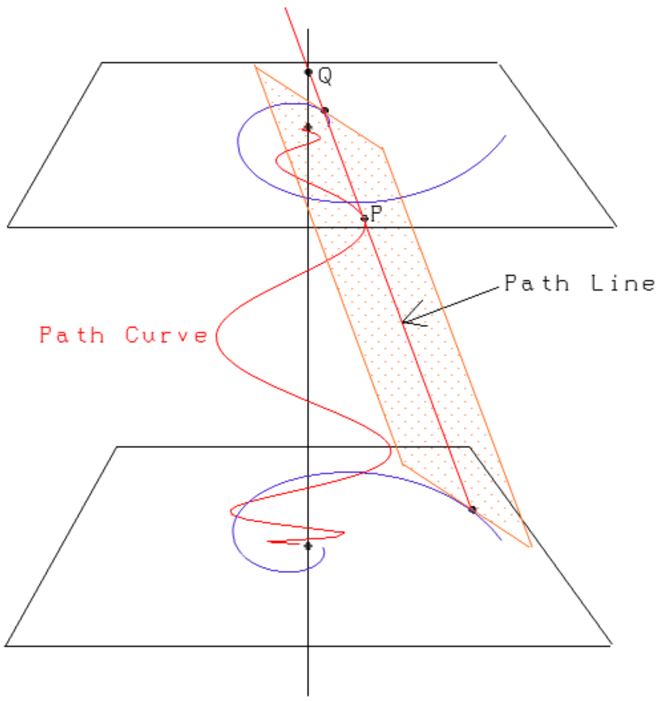

Figure 3

The line joining the points of contact is called the *path line*, and is evidently the line about which the plane is momentarily turning, as its tangent lines to the spirals move so as to remain tangential to the invariant spirals. The path line touches a path curve, so the path line is itself turning about its tangent point to that curve. That point is the pivot point of the plane, and is shown lying on the vertical profile of the egg surface concerned (which is woven of all the twisted path curves through a horizontal circle).

We now derive the most fundamental result for this work derived by Edwards, but by following a slightly different route from that in Reference 1. Figure 4 shows a plane β orthogonal to the paper intersecting the invariant planes in lines orthogonal to the points A and B, and the vertical axis in the point Q.

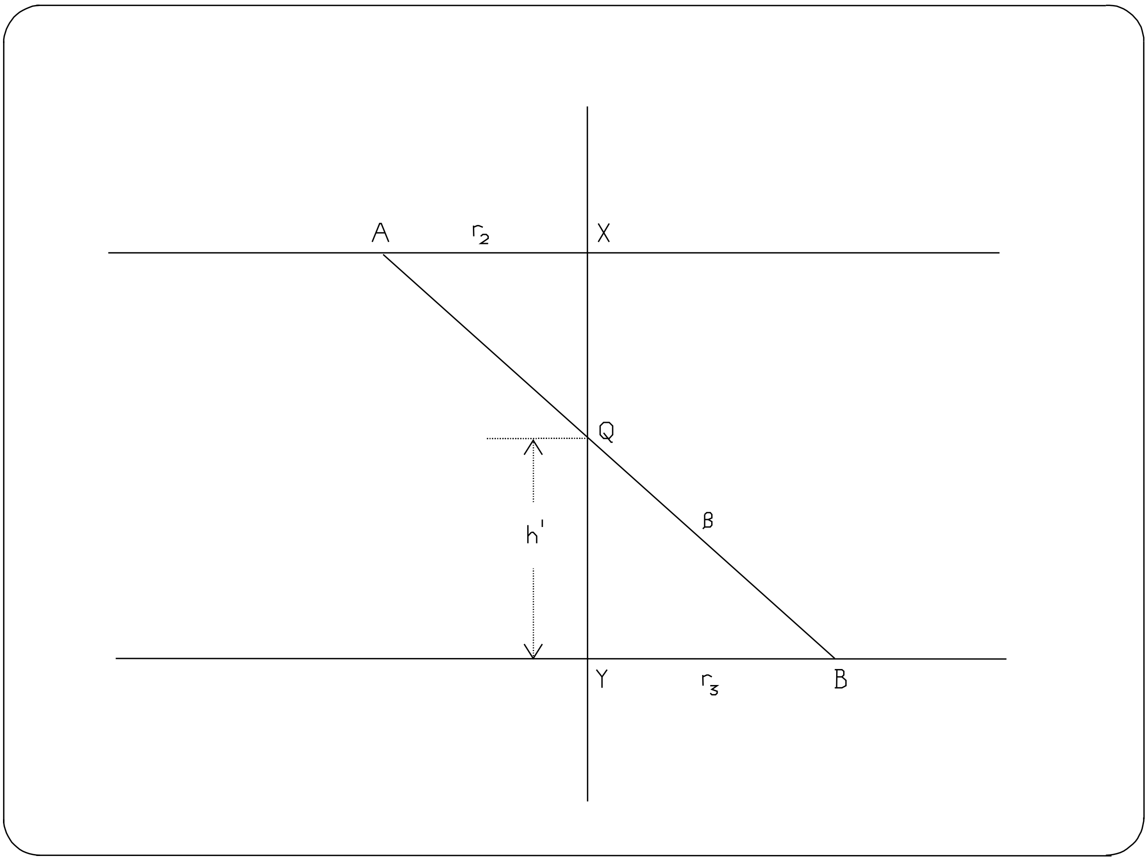

Figure 4

The lengths of the lines XA and YB are denoted by r_2 and r_3 respectively. The fractional height of Q (i.e. as a fraction of the height XY) is clearly

$$h' = \frac{r_3}{r_3 + r_2}$$

We will now relate this to the height h of the pivot point. The situation in the top invariant plane is as follows:

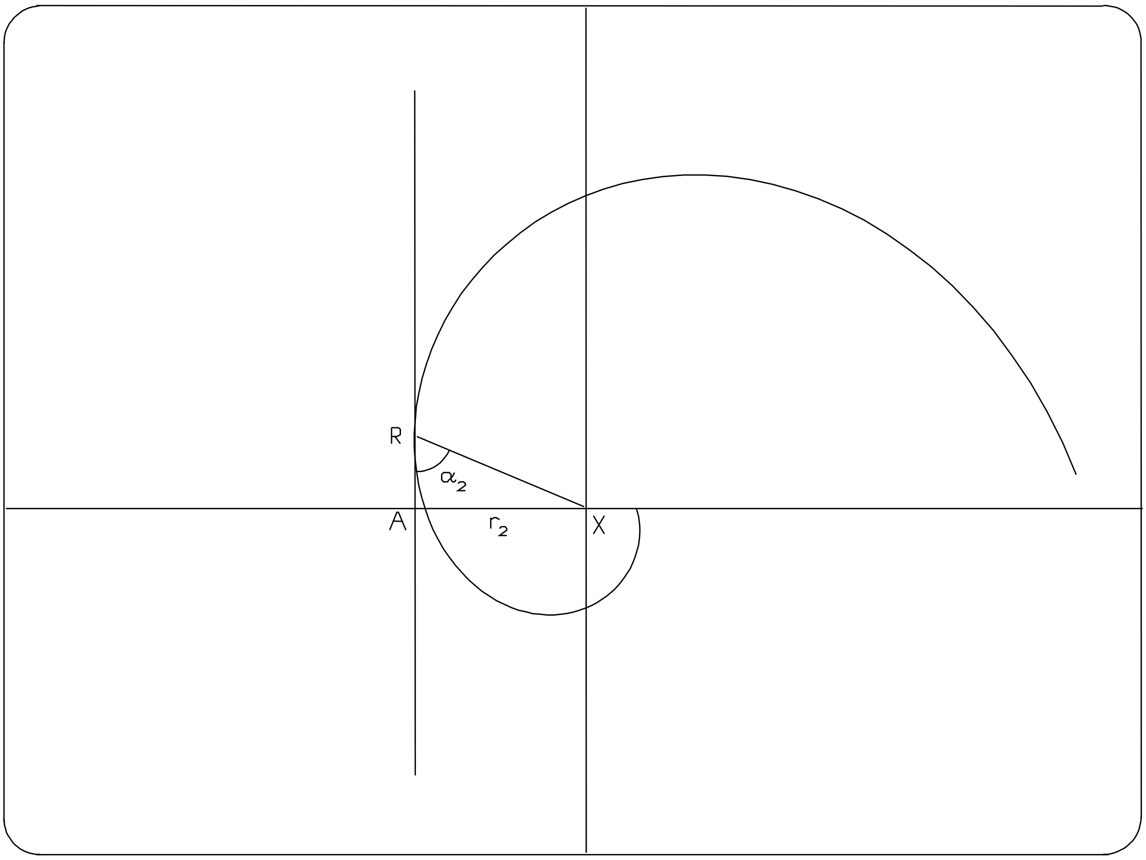

Figure 5

The plane β intersects the invariant plane in a line AR, and a spiral of the path curve system touches AR at R. The radius vector is at an angle α_2 to AR, and thus $XR=r_2/\sin\alpha_2$. The length of arc of the logarithmic spiral (a standard result) is

$$s = r \sec \alpha_2$$

and thus the rate of change of R along the curve wrt θ is

$$\frac{ds}{d\theta} = \frac{dr}{d\theta} \sec \alpha_2 = r \cot \alpha_2 \sec \alpha_2 = \frac{r}{\sin \alpha_2}$$

noting that $r=r_0e^{\theta \cot\alpha}$ is the equation of the spiral, so $dr/d\theta = r.\cot\alpha_2$.

In this case r=XR so the rate of change of s is

$$s' = \frac{r_2}{\sin^2 \alpha_2}$$

In the lower invariant plane we have a line BS touching a spiral at S, and for S we have

$$t' = \frac{r_3}{\sin^2 \alpha_3}$$

denoting the arc length by t. Now RS is the path line, and R and S are moving in opposite directions in their planes at the above rates, so the height of the point on RS which is instantaneously at rest - i.e. the pivot point - is by simple proportion

$$h = \frac{t'}{t'+s'} = \frac{r_3 \sin^2 \alpha_2}{r_3 \sin^2 \alpha_2 + r_2 \sin^2 \alpha_3} = \frac{\left(\frac{r_3}{r_2}\right)\sin^2 \alpha_2}{\left(\frac{r_3}{r_2}\right)\sin^2 \alpha_2 + \sin^2 \alpha_3}$$

Now from the previous expression for h' we have

$$\frac{r_3}{r_2} = \frac{h'}{1-h'} \qquad (1)$$

so substituting in the expression for h we get

$$h = \frac{h' \sin^2 \alpha_2}{h' \sin^2 \alpha_2 + (1-h') \sin^2 \alpha_3}$$

This important equation enables us to relate the height of a pivot point in a plane to the height of the intercept of that plane with the axis. Thus all planes meeting the axis in a fixed point Q have their pivot points in a fixed horizontal plane at a fractional height h. This enables us to find the horizontal profiles of a pivot form. Note that α_2 and α_3 are constant for a given transform **T**, so we may write the above equation as

$$h = \frac{k_2 h'}{k_2 h' + k_1 (1-h')} \qquad (2)$$

or

$$h' = \frac{k_1 h}{k_1 h + k_2 (1-h)} \qquad (3)$$

The definitions of k_1 and k_2 in Reference 1 are different but the result is identical.

We also need to find the radial distance of the pivot point from the axis, and for this we first require two simple results. From Figure 5 it is clear that

$$y = \mathrm{AR} = \frac{r_2}{\tan \alpha_2} \qquad (4)$$

and substituting in (1) for h' from (3) and simplifying we get

$$r_3 = \frac{k_1 h r_2}{k_2(1-h)} = \frac{h r_2 \sin^2 \alpha_3}{(1-h)\sin^2 \alpha_2} \qquad (5)$$

In Figure 6 we depict AR in the top plane as y_2 and BS in the bottom plane as y_3, noting they are on the parallel lines in which the plane we are transforming cuts those planes. Seen in plan view they are separated by a distance r_2+r_3. y is the coordinate of the pivot point P which is situated at some distance d from y_3. The axis is orthogonal to X. By simple proportion it is clear that

$$\frac{d}{r_2+r_3} = h \text{ , the fractional height of P.}$$

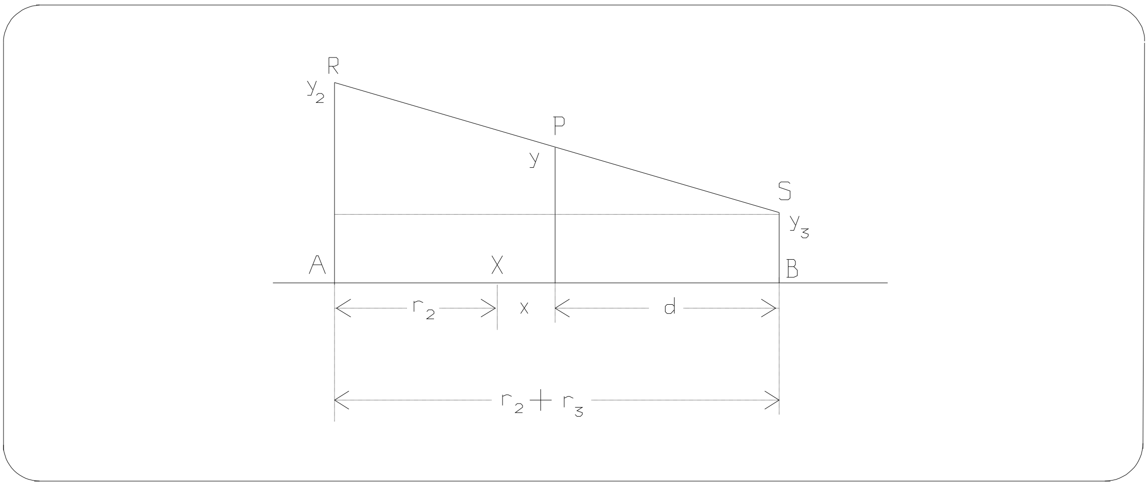

Figure 6

Again using similar triangles we find

$$\frac{y_2-y_3}{r_2+r_3} = \frac{y-y_3}{d}$$

$$\text{giving } \frac{d}{r_2+r_3} = h = \frac{y-y_3}{y_2-y_3}$$

$$\text{and hence } y = y_2 h + y_3(1-h)$$

Substituting for y_2 from (4) and using the similar result for y_3 gives

$$y = \frac{r_2 h}{\tan \alpha_2} + \frac{r_3(1-h)}{\tan \alpha_3}$$

and now substituting for r_3 from (5) and simplifying we get

$$y = r_2 h \frac{\sin\alpha_2 \cos\alpha_2 + \sin\alpha_3 \cos\alpha_3}{\sin^2\alpha_2} \tag{6}$$

Clearly x = r_3-d = r_3(1-h)-hr_2, and again substituting for r_3 from (5) gives

$$x = h r_2 \frac{\sin^2\alpha_3 - \sin^2\alpha_2}{\sin^2\alpha_2} \tag{7}$$

Finally combining (6) and (7) using Pythagoras' Theorem and wielding some trigonometry gives

$$r = r_2 h \frac{\sin(\alpha_2 + \alpha_3)}{\sin^2\alpha_2} = r_2 h k \tag{8}$$

defining k accordingly (which incidentally equals the definition of k_1 in Reference 1). This simple result enables the radial distance r of the pivot point from the axis to be calculated from the perpendicular distance r_2 from the axis to the line in which the pivotting plane intercepts the top invariant plane. Equations (2), (3) and (8) are what we need to transform any surface, and are remarkably simple considering the complexity of the process and the striking results we will obtain later.

Pivot Transforms of Vortices

We confine ourselves here to the transforms of a particular species of vortex. We illustrated egg-shaped path curves above where the logarithmic spirals in the invariant planes wound in opposite senses, but if they wind in the same sense then the path curve is a vortex form. We take the case where the upper invariant plane is at infinity. The vertical profile of such a vortex is obtained by setting θ=0 and imposing cartesian coordinates which forces the top invariant plane to be at infinity. Then a cartesian point {x,y,z,1} transforms to $\{\Phi x/\Lambda_3, \Phi y/\Lambda_3, \Lambda_2 z/\Lambda_3, 1\}$. Since the ratio if x and y is constant we indeed have a vertical profile. If $\Lambda_2/\Lambda_3>1$ then z tends towards infinity as the transformation is repeated since $z_n=z_0(\Lambda_2/\Lambda_3)^n$, and if $\Phi/\Lambda_3>1$ then the radius $r_n=r_0(\Phi/\Lambda_3)^n$ increases towards infinity as well. This gives us the kind of vortex we wish to study. If μ is defined by $(\Lambda_2/\Lambda_3)^{\mu} = \Phi/\Lambda_3$ then we obtain for the equation of the profile

$$z^{\mu} = z_0^{\mu}\left(\frac{\Lambda_2}{\Lambda_3}\right)^{\mu n} = z_0^{\mu}\left(\left\{\frac{\Lambda_2}{\Lambda_3}\right\}^{\mu}\right)^{n} = z_0^{\mu}\left(\frac{\Phi}{\Lambda_3}\right)^{n} = \frac{z_0^{\mu} r}{r_0}$$

$$\text{giving } r = \frac{r_0}{z_0^{\mu}} z^{\mu} = W z^{\mu}$$

We see that μ is constant for a given transformation, and W is a parameter determining the size of the vortex. Note that the values of Φ and Λ_j we have just used are not those of **T** but are for a different transformation defining the type of vortex we wish to study. In fact we do not invoke this second transformation at all, only the vortex as a static surface.

To find the cone of contact with its vertex at G (see Figure 7) we use the fundamental fact of projective geometry that the vortex may be constructed from its tangents by joining corresponding points of two projective ranges on the lines OC and the horizontal line OH through O. These ranges are related by a multiplier such that

$$OH = OG^{\mu}$$

where H is the point on OH corresponding to G in the projectivity. Then the radius of the circle in which the cone intersects the top invariant plane is, by similar triangles,

$$\begin{aligned} R &= OH\frac{GC}{OG} = OH\frac{e-u}{u} \\ &= W'\frac{e-u}{u}OG^{\mu} = W'\frac{e-u}{u}\left(\frac{OC\,u}{e}\right)^{\mu} \\ &= W\frac{e-u}{u}\left(\frac{u}{e}\right)^{\mu} \end{aligned}$$

where W is a size parameter which absorbs OC^{μ} as OC is constant.

A typical profile of such a vortex placed within the invariant tetrahedron of the egg transform **T** is shown below:

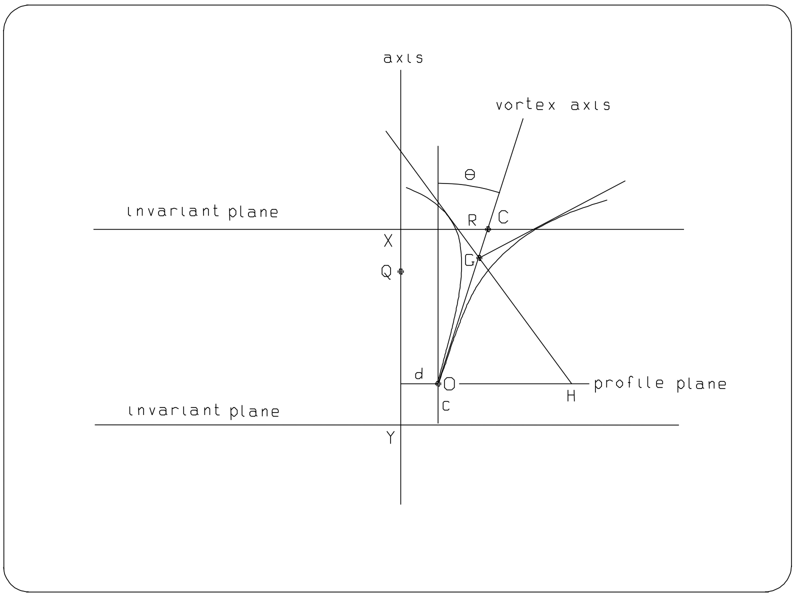

Figure 7

Notice that we have taken the axis skew to that of the egg. Its vertex is at O at a fractional height c above the lower plane and a radius d from the axis. There is a tangent cone in any point G on its axis, the vortex being such that it is circular. The pivot points of the tangent planes of this cone are also pivot points of the vortex. If we seek the pivot points in a particular profile plane related to Q then just two of the tangent planes in G contain Q and yield pivot points in that plane. Thus to obtain the complete profile we must move G up from O and find the pivot points of the two planes in Q for each position of G. For other profile planes we select different positions of Q.

If we take the profile plane through O, finding the corresponding height of G from (3) by setting h=c, then we obtain embryo-like forms as shown below (following Reference 1).

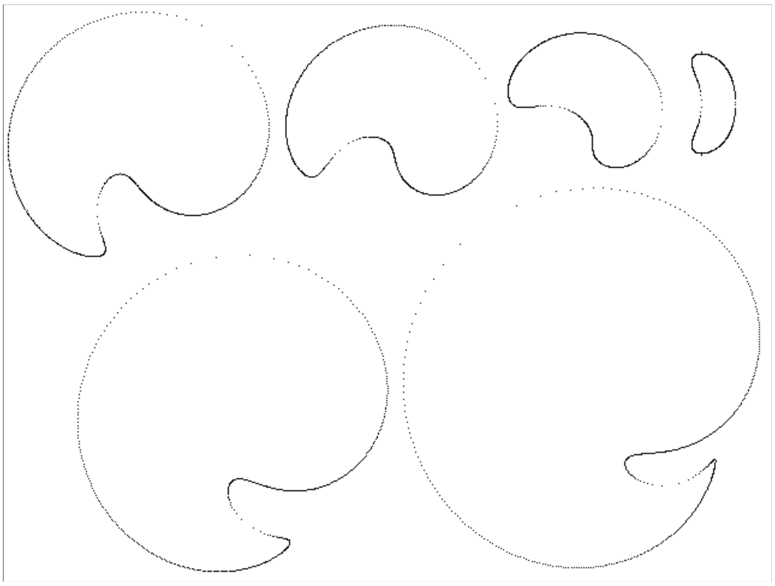

Figure 8

The parameters used for these forms were as summarised in the table below (taken from Reference 1), the first column being for the smallest form, and so on:

X_1,Y_1	0.06,0	-0.06,.38	-0.25,0.5	-0.5,0.55	-0.75,0.5	-0.94,.38
X_2,Y_2	0.3,0	0.3,0	0.3,0	0.3,0	0.3,0	0.3,0
c	0.3	0.3	0.3	0.3	0.3	0.3
Size	0.007	0.008	0.009	0.010	0.012	0.015
Vortex λ	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3

X_1,Y_1 are the coordinates of the intercept of the vortex axis with the top invariant plane.
X_2,Y_2 are the horizontal coordinates of O ($d^2=X_2^2+Y_2^2$).
The size factor is empirical.

We have not so far defined the parameter λ given above. It controls the overall shape of a path curve and is defined as

$$\lambda = -\frac{\log\phi - \log\Lambda_2}{\log\phi - \log\Lambda_3} \qquad (9)$$

If $\lambda = 1$ then for an egg the vertical profile is elliptic. If λ is increased above 1 the profile becomes more pointed at the top and blunt at the bottom, like a real egg. The three dimensional path curve surface is the surface of rotation of such a profile. If $\lambda = \infty$ then the surface becomes a cone. Reciprocal values of λ turn the egg upside-down. This is all fully explained in Reference 1. If $\lambda < 0$ then instead we have vortex profiles varying from conical ones when $\lambda = -\infty$ to flat spirals when $\lambda = -1$. Hence a value of –0.3 gives an intermediate case with the vertex at the bottom, such as was illustrated above.

Another fundamental parameter controls the spiralling of a path curve. It is defined as

$$\epsilon = \frac{\log \Lambda_2 - \log \Lambda_3}{2\theta} \tag{10}$$

When $\epsilon = \infty$ the curve does not spiral at all, but remains in a vertical plane. When $\epsilon = 0$ the path curves are horizontal circles (in the case treated here; in more general cases they are horizontal spirals). For other values of ϵ they spiral typically as shown in Figure 1. The sign of ϵ controls the sense.

Analysis of Vortex Case

The question then arises, what do the full three-dimensional versions of these forms look like? To explain the method of calculation it is necessary to analyse the vortex case in more detail. Figure 9 below shows the situation in the top invariant plane (say γ). The top left diagram shows the situation in γ, and at lower right the elevation view is shown. The circle in which the tangent cone to the vortex intersects γ is shown with centre $C(x_1,y_1)$ and radius R. The line joining the vertex G of the cone to Q intersects γ in $F(x_3,y_3)$. The two tangent planes in G which have pivot points in the profile plane intersect γ in lines through F tangential to the circle. The point in γ vertically above O is shown as $O'(x_2,y_2)$, and the similar projection of G is shown as G'. X is the origin in γ, and the x-coordinates are horizontal and the y_j are vertical. The slope of the tangent line shown is m in terms of these coordinates. The fractional height of G above O will be denoted by u.

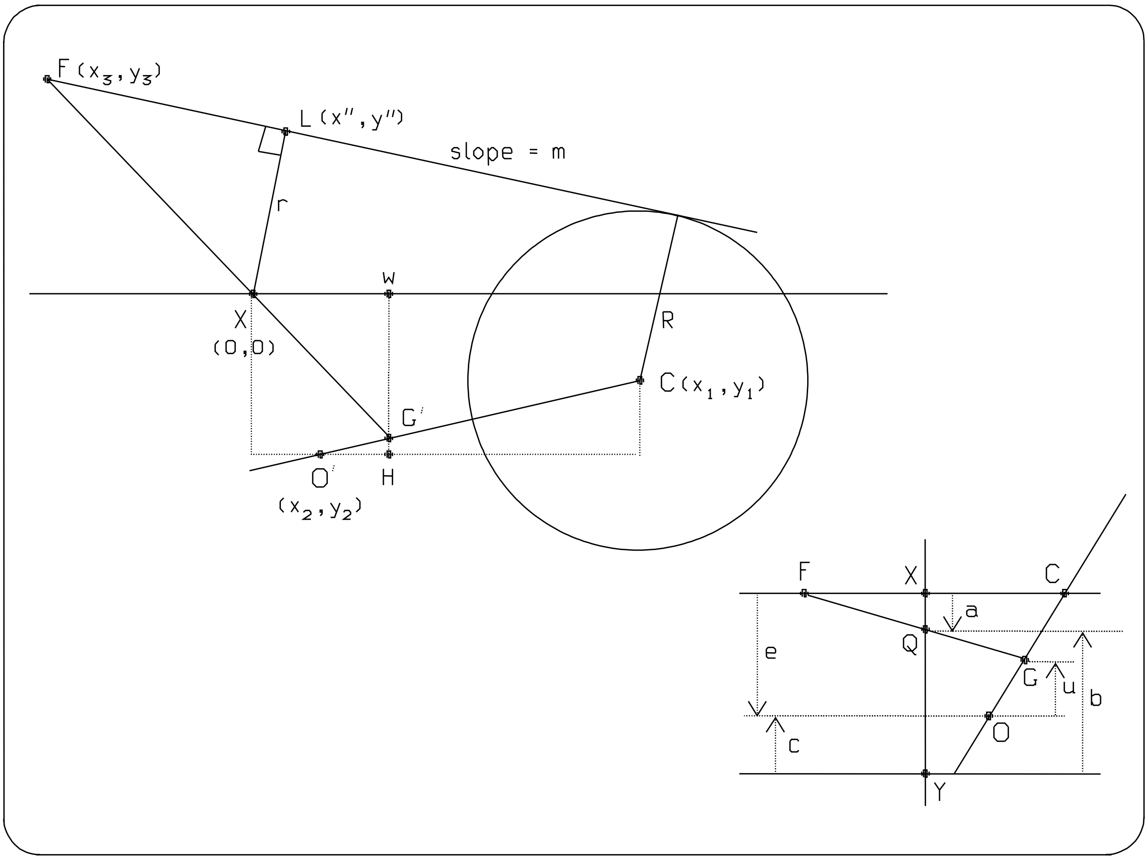

Figure 9

We will now derive expressions for x_3 and y_3.

By simple proportion we have

$$\frac{\mathrm{OG}}{\mathrm{OC}} = \frac{u}{e} = \frac{\mathrm{O'G'}}{\mathrm{O'C}} = \frac{\mathrm{O'H}}{x_1 - x_2}$$

and thus

$$\mathrm{O'H} = \frac{u(x_1 - x_2)}{e}$$

Also

$$\frac{-x_3}{x_2 + \mathrm{O'H}} = \frac{\mathrm{FX}}{\mathrm{XG'}} = \frac{a}{e - a - u}$$

giving

$$x_3 = \frac{a\left[ex_2 + u(x_1 - x_2)\right]}{e\left[a - e + u\right]} \tag{11}$$

Again

$$\frac{\mathrm{HG'}}{y_1 - y_2} = \frac{\mathrm{O'G'}}{\mathrm{O'C}} = \frac{u}{e}$$

giving

$$\mathrm{WG'} = -\,y_2 - \mathrm{HG'} = -\,y_2 - \frac{u(y_1 - y_2)}{e} = -\frac{\left[uy_1 + (e-u)y_2\right]}{e}$$

Now

$$\frac{-\,y_3}{\mathrm{WG'}} = \frac{\mathrm{FX}}{\mathrm{XG'}} = \frac{a}{e-a-u}$$

so

$$y_3 = \frac{a\left[ey_2 + u(y_1 - y_2)\right]}{e\left[a-e+u\right]} \tag{12}$$

We now need to find the coordinates (x,y) of the pivot point. Noting that $m<0$ and $x_3<0$ in Figure 9, we have

$$\begin{aligned}
&y'' = y_3 - m\,x_3 + m\,x'' \\
&x'' = -m\,y'' \\
&\text{so}\; y'' + m^2 y'' = y_3 - m\,x_3 \\
&\text{i.e.}\; y'' = \frac{y_3 - m\,x_3}{1+m^2}
\end{aligned}$$

Now y" and x" are the components of the normal XL=r from X to the line in which the plane whose pivot point P we seek meets γ. Thus since we can apply (8) to r we can also apply it to its components, so that for P we have

$$\begin{aligned}
y &= \frac{kh(y_3 - mx_3)}{1+m^2} \\
x &= -my
\end{aligned} \tag{13}$$

This solves our problem in principle, enabling the pivot transform of the vortex to be calculated.

<u>Practical Results</u>

Initially the author built a wooden model of one case, but this was a very time consuming approach and instead it seemed desirable to construct “photographs” graphically of what such models would look like. For each pixel on the computer screen this requires us to find the orientation of the tangent plane to the surface nearest to the viewer at that point, from which the illumination intensity may be calculated depending upon the direction of illumination. While the maths for that is not trivial, the hardest part was solving the transcendental equations which arise for the points of intersection of the lines orthogonal to the screen with the pivot transform. A bisection approach was adopted, but that had to be made quite sophisticated as the roots can be very close together in some circumstances. This required an analysis of the slope of the curve as part of the bisection algorithm, together with logical tests for special cases.

The derivation of the formulae used to calculate the tangent planes to the surface is given in Annex 1, and the bisection equations are given in Annex 2. The appearance of blemishes in some pictures is discussed in Annex 2.

Coloured prints of the results are contained in the separate file figs.pdf, as tabulated below.

Figure number	Description
Figure 1	Three dimensional versions of Figure 8
Figure 2	First three of above cases from various views
Figure 3	Last three of above cases from various views
Figure 4	Effect of varying vortex λ with c=0
Figure 5	Cases with vortex axis intersecting bud axis
Figure 6	Cases with vortex axis through X
Figure 7	Axis vertical & close to bud axis; λ varying
Figure 8	Cases with vortex axis meeting orth line to axis
Figure 9	Large d, axis at –40 & at azimuth 160
Figure 10	Effect of varying vortex axis angle to vertical
Figure 11	Effext of varying d
Figure 12	Effect of varying azimuth of vortex axis tilt
Figure 13	Effect of varying vortex size
Figure 14	Effect of varying vortex λ
Figure 15	Effect of varying bud λ
Figure 16	Effect of varying fractional height of O
Figure 17	Effect of varying bud ϵ

The first three show the results for the embryo case of Reference 1.
Figures 10 to 16 show the results when varying one parameter at a time. All started from the base parameters as follows:

bud $\lambda = 1$
bud $\epsilon = 1$
Vortex $\lambda = -0.5$
Fractional height of O = 0.5
Vortex axis vertical & very close to bud axis.

Annexes

These are in separate documents.

1. Tangent Planes to Surfaces.

2. Bisection Method.

3. Construction of Pivot Transforms.

References

1. "The Vortex of Life", Lawrence Edwards
   Floris Press, Edinburgh 1993.

2. "Algebraic Projective Geometry" by Semple and Kneebone
   Oxford University Press, Oxford 1952.