PRACTICAL PATH CURVE CALCULATIONS

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Aim

The aim of this paper is to derive some formulae and results useful for practical work with path curves e.g. for calculation or computer programming. The basic theory is covered in Edwards 1982 and 1993, to which the reader is referred for the concept of path curves.

Introduction

In brief a path curve is the locus of a point which is repeatedly moved by a linear (projective) transformation, or dually the cuspidal edge of the developable arising when a plane is repeatedly transformed in the same way. Such a transformation leaves a tetrahedron invariant, which in the special case treated in this article consists of two real horizontal parallel planes, a real vertical axis and the real line at infinity in which the two planes meet. In addition there are two real invariant points where the axis meets the planes. The remaining two planes, four sides and two invariant points of the tetrahedron are imaginary. These matters are dealt with in Edwards (all). The reader may also consult www.nct.anth.org.uk.

Basic Formulae

We will first derive the basic formulae. Figure 1 shows the invariant tetrahedron with X and Y as the real invariant points. Two logarithmic spirals are shown in the top and bottom planes together with a shaded plane which revolves about the axis. It meets the spirals in two points A and B (apart from other points in each plane not shown), and the two lines AY and XB meet in a point P lying on the spiral path curve shown. As the plane rotates P describes the path curve. Two cases arise: either the spirals wind round the axis in the same sense in which case the path curve is a vortical spiral, or else they do so in opposite senses giving an egg-shaped curve such as that illustrated. If the curve is rotated about the axis then it creates an egg-shaped surface; hence the name. All this will be taken without proof as our starting point, building on the work of George Adams and Lawrence Edwards. Some of their formulae will be reproduced as we go along.

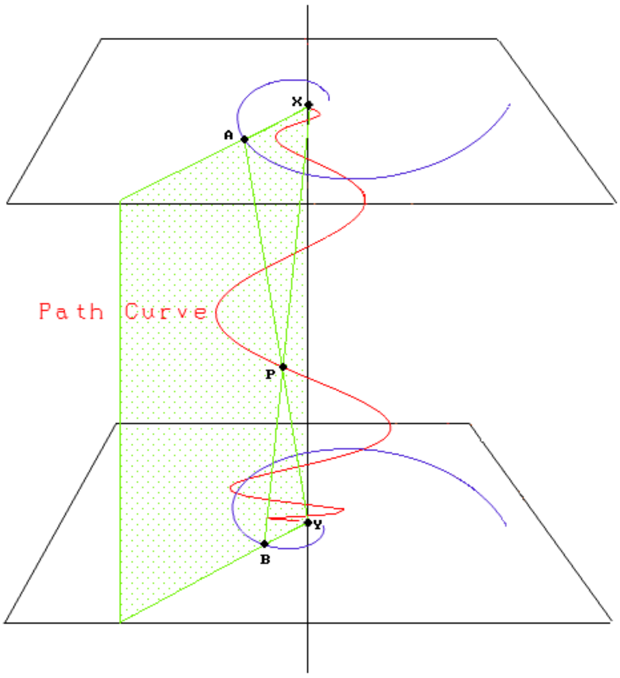

Figure 1

We will need the well known equation for a logarithmic spiral:

$$r = r_0 e^{\theta \cot \tau} \tag{1}$$

where r is the radius of a point from the centre, r_0 is the initial radius when the angle θ turned through is zero, and τ is the (constant) angle between the radius and tangent at any point on the curve.

In Figure 1 let the radial distance of P from the axis be r, its height above the bottom plane be h, and the separation of the planes be H. Let XA = u and YB = v, so that from (1)

$$u = u_0 e^{-\theta \cot \psi} \quad and \quad v = v_0 e^{\theta \cot \phi} \tag{2}$$

defining ψ and Φ accordingly, which are constant. The reason for the negative sign is that we will consider the case when the spirals wind in opposite directions to yield eggs, so u must decrease with θ and v increase or *vice versa*. By simple proportion the radius r is given by

$$r = \frac{uv}{u+v} \tag{3}$$

and the height by

$$h = \frac{Hv}{u+v} \tag{4}$$

Thus when $\theta = 0$, (3) and (4) give r_0 and h_0 in terms of u_0 and v_0, so (3) and (4) may be solved for u_0 and v_0 giving

$$u_0 = \frac{r_0 H}{h_0} \text{ and } v_0 = \frac{r_0 H}{H - h_0} \tag{5}$$

Substituting (2) in (3) gives

$$r = \frac{u_0 v_0 e^{\theta(\cot\phi - \cot\psi)}}{u_0 e^{-\theta \cot\psi} + v_0 e^{\theta \cot\phi}} = \frac{u_0 v_0}{u_0 e^{-\theta \cot\phi} + v_0 e^{\theta \cot\psi}}$$

and now substituting for u_0 and v_0 from (5) gives

$$r = \frac{r_0 H}{\left(H - h_0\right) e^{-\theta \cot\phi} + h_0 e^{\theta \cot\psi}} \tag{6}$$

Similarly, substituting (2) and (5) in (4) gives

$$h = \frac{H}{\frac{H - h_0}{h_0} e^{-\theta(\cot\psi + \cot\phi)} + 1} \tag{7}$$

Edwards defines two important parameters which describe path curves as follows:

1. If the ratio $u/u_0 = b$ in the top plane when $\theta=2\pi$ i.e. after one revolution, and similarly $v/v_0 = c$ in the bottom plane so that b and c are the multipliers of the geometric series produced by the spirals, then the parameter λ is defined as the ratio $-\frac{\log b}{\log c}$. This is illustrated in Figure 2.

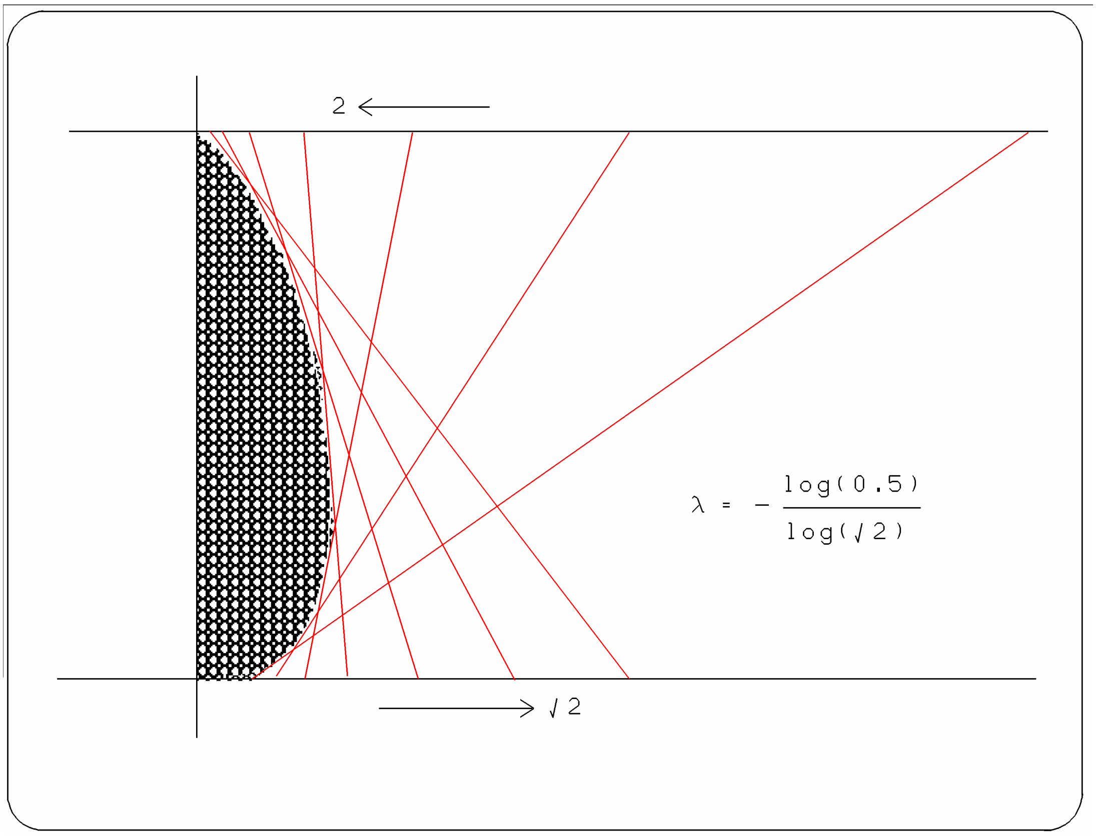

Figure 2

Now from (2) we have

$$b = \frac{u_0 e^{-2\pi \cot \psi}}{u_0} = e^{-2\pi \cot \psi} \text{ and similarly } c = e^{2\pi \cot \phi} \text{ which gives}$$

$$\lambda = -\frac{\cot \psi}{\cot \phi} \tag{8}$$

2. The spiralling is controlled by the parameter ε defined as half the sum of log$b$ and log$c$ which is

$$\frac{1}{2}\left(2\pi \cot \phi - 2\pi \cot \psi\right)$$

However, ε is also defined <u>per radian turned</u>, so we remove the factors of 2π to give

$$\varepsilon = \frac{1}{2}\left(\cot \phi - \cot \psi\right) \tag{9}$$

Solving (8) and (9) for Φ and ψ gives

$$\cot \phi = \frac{2\varepsilon}{1+\lambda} \qquad \cot \psi = \frac{-2\,\varepsilon\lambda}{1+\lambda} \tag{10}$$

Finally substituting these in (6) and (7) we get

$$r = \frac{r_0 H}{\left(H - h_0\right) e^{\frac{-2\varepsilon\theta}{1+\lambda}} + h_0 e^{\frac{2\lambda\varepsilon\theta}{1+\lambda}}} \tag{11}$$

$$h = \frac{H}{\frac{H - h_0}{h_0} e^{-2\varepsilon\theta} + 1} \tag{12}$$

These are the most general equations (for this type of tetrahedron), without special assumptions about the separation of the planes or the location of (r_0, h_0), and are most useful in practice. Notice that h is independent of r_0, but r <u>is</u> dependent upon h_0. If $\lambda > 0$ the exponents of the two terms in the denominator of (11) are of opposite sign and so as θ increases dominance moves from the one to the other, which is why an egg form is produced as the denominator does not tend to zero. If on the other hand $\lambda < 0$ both exponents have the same sign and the denominator changes from zero to infinity, yielding a vortex. Note that although (11) and (12) were derived for an egg, they apply also to vortices as assuming a positive exponent for u in (2) and defining ϵ accordingly as $\frac{1}{2}\left(\log c - \log b\right)$ yields the same final result.

Another useful equation is obtained by dividing r by h which gives

$$\frac{r}{h} = \frac{r_0}{h_0} e^{\frac{-2\lambda\varepsilon\theta}{1+\lambda}} \tag{13}$$

If $\varepsilon=0$ then $r=r_0$ and $h=h_0$ which means the path curves are horizontal circles as the choice of r_0 and h_0 is arbitrary. If $\varepsilon=\infty$ then $\theta=0$ to avoid infinite quantities, and so the path curves are vertically situated profiles, but the equations are singular as $\varepsilon\theta$ is indeterminate. We will see how to work with this below.

<u>Cosmic Vortex</u>

If one of the real invariant planes is at infinity then we have what may be called a *cosmic vortex* which has proved important in practice, particularly in the study of water vortices. In this case the vertex of the vortex is at infinity so $H = \infty$ in (11) and (12), giving the simple exponentials

$$r = r_0 e^{\frac{2\varepsilon\theta}{1+\lambda}} \quad h = h_0 e^{2\varepsilon\theta} \tag{14}$$

i.e. parametric equations in θ. The equation for r is a case of equation (1), so the orthogonal projection of the curve onto any horizontal plane is a logarithmic spiral with

$$\cot \tau = \frac{2\varepsilon}{1+\lambda}$$

Although the equation for h is similar we do not obtain a spiral because h is not a radial quantity; instead we get an exponential curve.

An alternative form of equations (14) is obtained by eliminating $e^{2\varepsilon\theta}$

$$h = \left[\frac{h_0}{r_0^{1+\lambda}}\right] r^{1+\lambda} \quad \text{or} \quad r = \left[r_0 h_0^{\frac{-1}{1+\lambda}}\right] h^{\frac{1}{1+\lambda}} \tag{15}$$

which can be useful when θ is unknown or unimportant or epsilon=∞, but r_0 and h_0 are known.

If $\frac{1}{1+\lambda}>0$ then r=0 when h=0 so the vertex is accessible and the curve tends to an invariant line at infinity, which is what Lawrence Edwards referred to as an “airy vortex”.

If $\frac{1}{1+\lambda}<0$ then $r\rightarrow\infty$ as $h\rightarrow 0$ and the vertex is at infinity, giving what Edwards refered to as a “watery vortex”.

Finding λ and ε from Two Known Points

We can solve (12) for ε:

$$\varepsilon = \frac{1}{2\theta}\log\left[\frac{h(H-h_0)}{h_0(H-h)}\right] \tag{16}$$

If now we multiply the numerator and denominator of (11) and (12) by $e^{\frac{2\varepsilon\theta}{1+\lambda}}$ and solve for λ we get

$$\lambda = \frac{2\varepsilon\theta}{\log\left[\frac{r}{Hr_0}\left(H-h_0+h_0 e^{2\varepsilon\theta}\right)\right]} - 1 \tag{17}$$

This does not solve the most general problem (when the tetrahedron is unknown), but if the tetrahedron is known then knowledge of (r_0,h_0), (r,h) and θ gives λ and ε.

Equation of Profile

A very useful equation due to B. Christian (1979) gives the relationship between r and h when we are only concerned with the two-dimensional profile of a path curve, or what amounts to the same thing, when $\varepsilon = \infty$. First we normalise the coordinates by setting H=2 and h_0=1 so that from (12) we get

$$e^{-2\varepsilon\theta} = \frac{2-h}{h} \tag{18}$$

and substituting this in (11) together with H=2 and h_0=1 we get

$$r = \frac{2\mathrm{r}_0}{\left(\frac{2-h}{h}\right)^{\frac{1}{1+\lambda}} + \left(\frac{2-h}{h}\right)^{\frac{-\lambda}{1+\lambda}}} \tag{19}$$

Now we change the reference for the height so that it varies from –1 to +1 between the invariant planes, and thus replace h by 1+ $\bar{h}$ in (19) to give

$$r = \frac{2\mathrm{r}_0}{\left(\frac{1-\bar{h}}{1+\bar{h}}\right)^{\frac{1}{1+\lambda}} + \left(\frac{1-\bar{h}}{1+\bar{h}}\right)^{\frac{-\lambda}{1+\lambda}}} = \frac{2\mathrm{r}_0(1+\bar{h})^{\frac{1}{1+\lambda}}(1-\bar{h})^{\frac{\lambda}{1+\lambda}}}{(1-\bar{h})^{\frac{1}{1+\lambda}}(1-\bar{h})^{\frac{\lambda}{1+\lambda}}+(1+\bar{h})}$$

i.e.

$$r = r_0(1+\bar{h})^{\frac{1}{1+\lambda}}(1-\bar{h})^{\frac{\lambda}{1+\lambda}} \tag{20}$$

which is an elegant result when normalised coordinates are satisfactory. It is particularly useful when $\varepsilon=\infty$ as then equations (11) and (12) are singular.

If we are working in three dimensions then we may use Christian's formula for r, and a separate formula for θ. Replacing h by 1+ $\bar{h}$ in (18) and re-arranging we have

$$\bar{h} = \frac{1-e^{-2\varepsilon\theta}}{1+e^{2\varepsilon\theta}} = \frac{e^{\varepsilon\theta}-e^{-\varepsilon\theta}}{e^{\varepsilon\theta}+e^{-\varepsilon\theta}} = \tanh(\varepsilon\theta)$$

so

$$\theta = \frac{\tanh^{-1}\bar{h}}{\varepsilon} \tag{21}$$

giving us in effect parametric equations in $\bar{h}$. This assumes $\theta=0$ when $r=r_0$.

Maximum Radius

In Figure 2 the egg profile has a maximum radius which is simply related to λ. Differentiating (20) we get

$$\frac{dr}{d\bar{h}} = r_0\left[\frac{(1+\bar{h})^{\frac{1}{1+\lambda}}(1-\bar{h})^{\frac{\lambda}{1+\lambda}}}{(1+\lambda)(1+\bar{h})} - \frac{\lambda(1-\bar{h})^{\frac{\lambda}{1+\lambda}}(1+\bar{h})^{\frac{1}{1+\lambda}}}{(1+\lambda)(1-\bar{h})}\right] = \frac{r}{1+\lambda}\left[\frac{1}{1+\bar{h}}-\frac{\lambda}{1-\bar{h}}\right] \tag{22}$$

which is zero when

$$\bar{h} = \frac{1-\lambda}{1+\lambda} \tag{23}$$

Thus for $\lambda=1$ (an ellipse) $\bar{h}=0$ as we expect. Replacing $\bar{h}$ by (h-1) and reverting to the original coordinates we find

$$h = \frac{H}{1+\lambda} \tag{24}$$

so for λ=2 the maximum radius is at one third of H, and so on, which gives an intuitive feel for λ and the reason for it being defined as it is, as well as a quick way of assessing it visually. (24) makes it clear that for a given λ all egg path curves regardless of r_0 and h_0 have their maximum radius at the same relative height.

Substituting for $\bar{h}$ in (20) we find the maximum radius is

$$r_{MAX} = \frac{2\bar{r}_0 \lambda^{\frac{1}{1+\lambda}}}{1+\lambda} \tag{25}$$

in normalised coordinates, which we have indicated by writing $\bar{r}_0$ = radius at $\bar{h} = 0$ or h=H/2. Note this only works for eggs, as if λ<0 then $|\bar{h}|>1$ which means it does not lie between the invariant planes.

Tangents to Path Curves

Recall that a path curve is also obtained by transforming planes, and the self-dual nature of the transform means that the cuspidal edge determined by the planes can be made to coincide with the point-wise path curve of Figure 1. Figure 3 shows the dual construction where the shaded plane touches two spirals, and as it is momentarily turning about its points of contact with them, the line joining those points is tangential to the path curve and is known as the path line. We will now relate this, in view of the duality involved, with the point-wise construction and thus see how to find the tangent. We follow Edwards (1982) in this process.

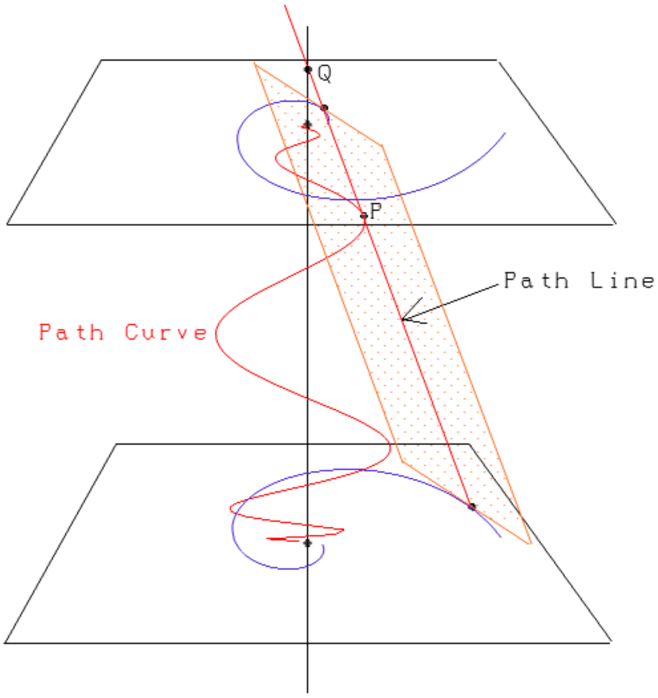

Figure 3

In Figure 4 we show a point P on the curve constructed as in Figure 1 with XM_2 meeting YM_1 in P. We draw the tangents to the spirals at M_1 and M_2 as shown. In the bottom plane we draw the line through Y parallel to the tangent in the top plane, which meets the tangent at M_2 in U_2. In the top plane we construct U_1 similarly using the line XU_1 paralell to M_2U_2. We will show that U_1U_2 is the path line which passes through P.

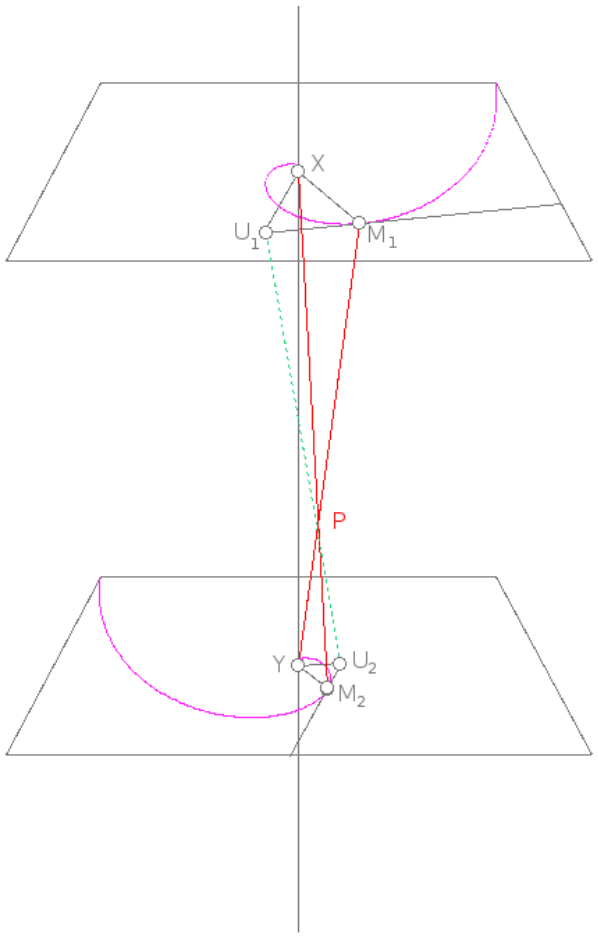

Figure 4

Figure 5 shows the top and bottom planes of Figure 4 superimposed. If we draw the line l_1 through U_1 at an angle ψ to XU_1 then l_1 is tangential to a spiral of the family in the top plane passing through U_1 (as they all have ψ as the angle between their radii and tangents, taking ψ as positive but opposite in sense to ϕ). Similarly we draw the line l_2 through U_2 in the bottom plane at an angle ϕ to YU_2, which must be tangential to a spiral of the bottom plane. The angle YM_2U_2 = ϕ since M_2U_2 is tangential to a spiral at M_2, and as M_2U_2 is parallel to XU_1 the angle M_1XU_1 = ϕ (alternate to YM_2U_2). The angle XM_1U_1 = ψ as M_1U_1 is tangential to a spiral at M_1, so M_1U_1 is at an angle ϕ to l_1 as shown (interior and exterior angles of triangle XM_1U_1). Similarly M_2YU_2 = ψ and l_2 is at an angle ψ to M_2U_2. As M_1U_1 and YU_2 are parallel it follows that l_1 is parallel to l_2. But in that case l_1 and l_2 are lines like the two tangents shown in Figure 3, so we have the dual construction combined with the original.

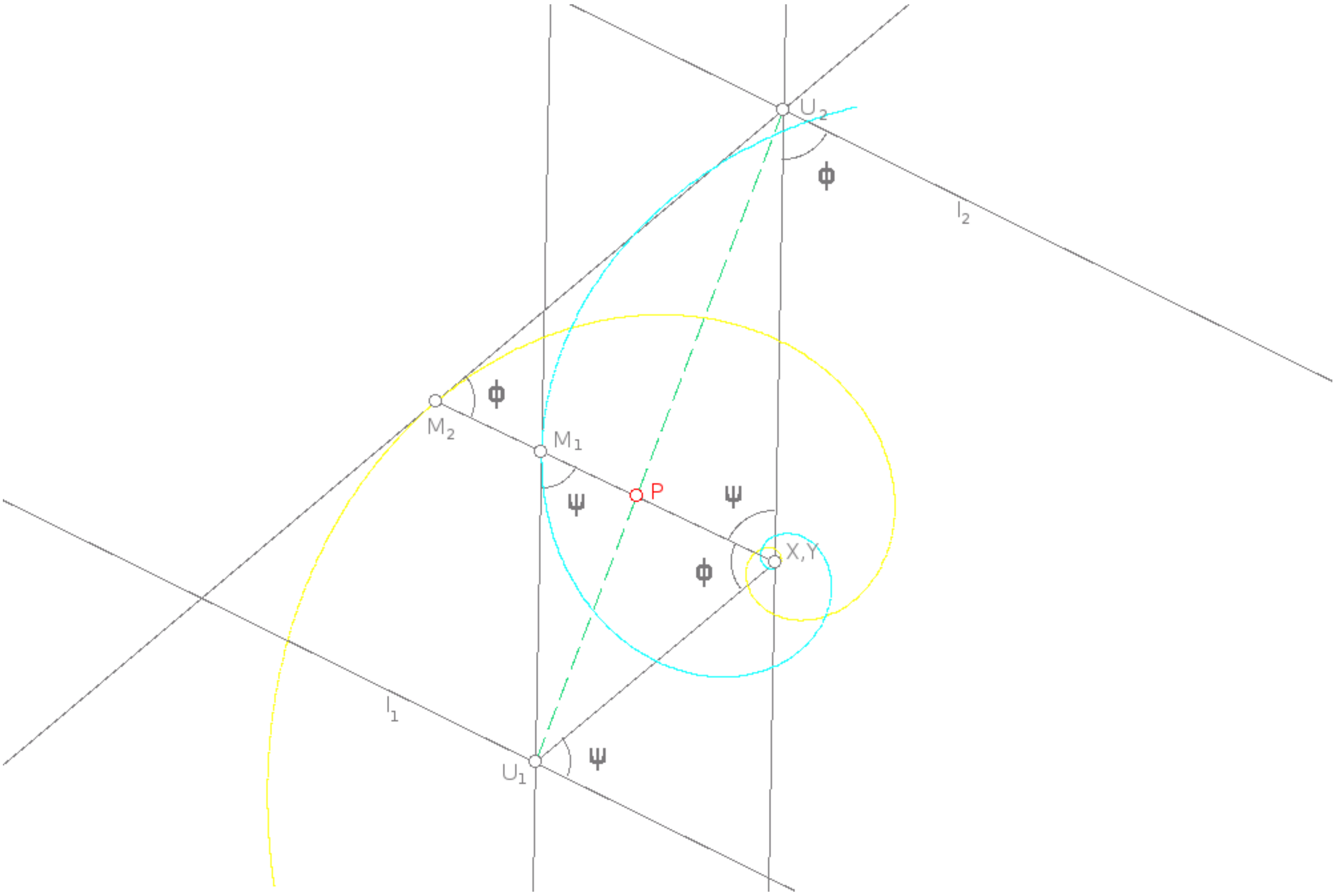

Figure 5

Now U_1 and U_2 both lie in the plane (U_1 M_1 Y) (c.f. Figure 4) so (U_1 M_1 Y) meets the plane (l_1,l_2) in the line U_1U_2, and similarly it meets the plane (U_2 M_2 X) in U_1U_2, so it follows that (U_1 M_1 Y) meets (U_2 M_2 X) in U_1U_2. Both those planes contain P, so the path line U_1U_2 passes through P and hence is the tangent at P (since P lies on the path curve and the path line is by definition a tangent to the curve meeting it in only one point).

We are now in a position to calculate. Referring to Figure 5, using the sine rule in triangle XU_1M_1 we have

$$\frac{XM_1}{\sin(\phi+\psi)} = \frac{XU_1}{\sin\psi} \quad \text{so} \quad XU_1 = XM_1\frac{\sin\psi}{\sin(\phi+\psi)}$$

and similarly

$$YU_2 = YM_2\frac{\sin\phi}{\sin(\phi+\psi)}$$

which are the radii we want. Using equations (10), and (5) for (r,h) instead of (r_0,h_0), we have

$$XU_1 = \frac{rH}{h}\frac{\sin\psi}{\sin(\phi+\psi)} = \frac{rH}{h}\frac{\tan\psi\sec\phi}{\tan\phi+\tan\psi} = \frac{rH}{h}\frac{\frac{1+\lambda}{2\varepsilon\lambda}\sec\phi}{\frac{1+\lambda}{2\varepsilon}+\frac{1+\lambda}{2\varepsilon\lambda}} = \frac{rH}{h(\lambda+1)}\sec\phi$$

noting that we have used ψ instead of $-\psi$ in the diagrams (as a positive quantity opposite in sense to ϕ, so we ignore the negative sign in (10). Similarly

$$YU_2 = \frac{rH\lambda}{(H-h)(\lambda+1)}\sec\psi$$

The angle $U_1XM_1 = \phi$, so XU_1 is at an angle $(\theta-\phi)$, assuming θ increases clockwise. Similarly $U_2YM_2 = \psi$ so YU_2 is at $(\theta+\psi)$. We conclude that the tangent is the join of the points

$$\left\{\frac{rH\sec\phi}{h(\lambda+1)},\ H,\ \theta-\phi\right\} \text{ and } \left\{\frac{rH\lambda\sec\psi}{(H-h)(\lambda+1)},\ 0,\ \theta+\psi\right\} \tag{26}$$

If the path curve winds in the opposite sense i.e. the senses of the two spirals are reversed, then the signs of ϕ and ψ must be reversed.

If ϕ and ψ are zero then we have for a two-dimensional path curve ($\varepsilon = \infty$) the join of

$$\left\{\frac{rH}{h(\lambda+1)},\ H\right\} \text{ and } \left\{\frac{rH\lambda}{(H-h)(\lambda+1)},\ 0\right\} \tag{27}$$

Projective Equations

Edwards (1982) gives the following equation for a two-dimensional path curve in terms of homogeneous coordinates (so far we have employed metric coordinates on the assumption that we are working with physical realisations of path curves):

$$x^u y^v z^w = k \tag{28}$$

A proof is given by Thomas (2001) (but presumably originally by Felix Klein, the discoverer of path curves) and in addition the following formula for three-dimensional path surfaces is derived:

$$x^u y^v z^w t^s = k \tag{29}$$

Two such surfaces intersect in a path curve.

Parametric Equations

The following parametric equations may be obtained for path curves (Thomas 2001):

$$x = k_1 t^{\alpha}, \quad y = k_2 t^{\beta}, \quad z = k_3 t^{\gamma} \qquad (30)$$

where k_1, k_2, k_3, α, β, γ are constants (real or complex) and (x,y,z,t) are the homogeneous coordinates. The Cartesian metric form is

$$x = k_1 z^{\alpha}, \quad y = k_2 z^{\beta} \qquad (31)$$

We have already met other parametric equations in (14) with θ as the parameter, and (20) combined with (21) with $\bar{h}$ as parameter.

Volume of a Cosmic Vortex

First we integrate along the axis between two heights. The element of volume is $\pi r^2 dh$, so the volume is

$$V = \pi \int_{h_1}^{h_2} r^2 \, dh$$

From (15) we thus get

$$V = \left[\frac{\pi r_0^2}{h_0^{\frac{2}{1+\lambda}}} \right] \int_{h_1}^{h_2} h^{\frac{2}{1+\lambda}} \, dh = \left[\frac{\pi r_0^2 (1+\lambda)}{(3+\lambda) h_0^{\frac{2}{1+\lambda}}} \right] \left(h^{\frac{3+\lambda}{1+\lambda}} \right)_{h_1}^{h_2}$$

This is invalid if $\lambda = -3$, so starting again we obtain

$$V = \left[\frac{\pi r_0^2}{h_0^{\frac{2}{1+\lambda}}} \right] (\log h)_{h_1}^{h_2}$$

Alternatively we may find the radial volume between two radii for a volume element of $2\pi rhdr$, and again using (15) we obtain

$$V = \left[\frac{2\pi h_0}{r_0^{1+\lambda}} \right] \int_{r_1}^{r_2} r^{2+\lambda} \, dr = \left[\frac{2\pi h_0}{(3+\lambda) r_0^{1+\lambda}} \right] \left(x^{3+\lambda} \right)_{r_1}^{r_2}$$

When $\lambda = -3$ we get

$$V = \frac{2\pi h_0}{r_0^{1+\lambda}} (\log r)_{r_1}^{r_2}$$

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